**Figure 1: An Example of Tele-parallel Geometry**.
Polar coordinates in the Euclidean plane underlie a Ricci Grid with unit vectors along its radii and circles. Radii and circles are auto-parallel curves in this geometry and ABCD thus form a rectangle. Different radii are tele-parallel and so are concentric circles. The fact that opposite sides AD and BC have different length is a characteristic feature in this geometry with torsion.
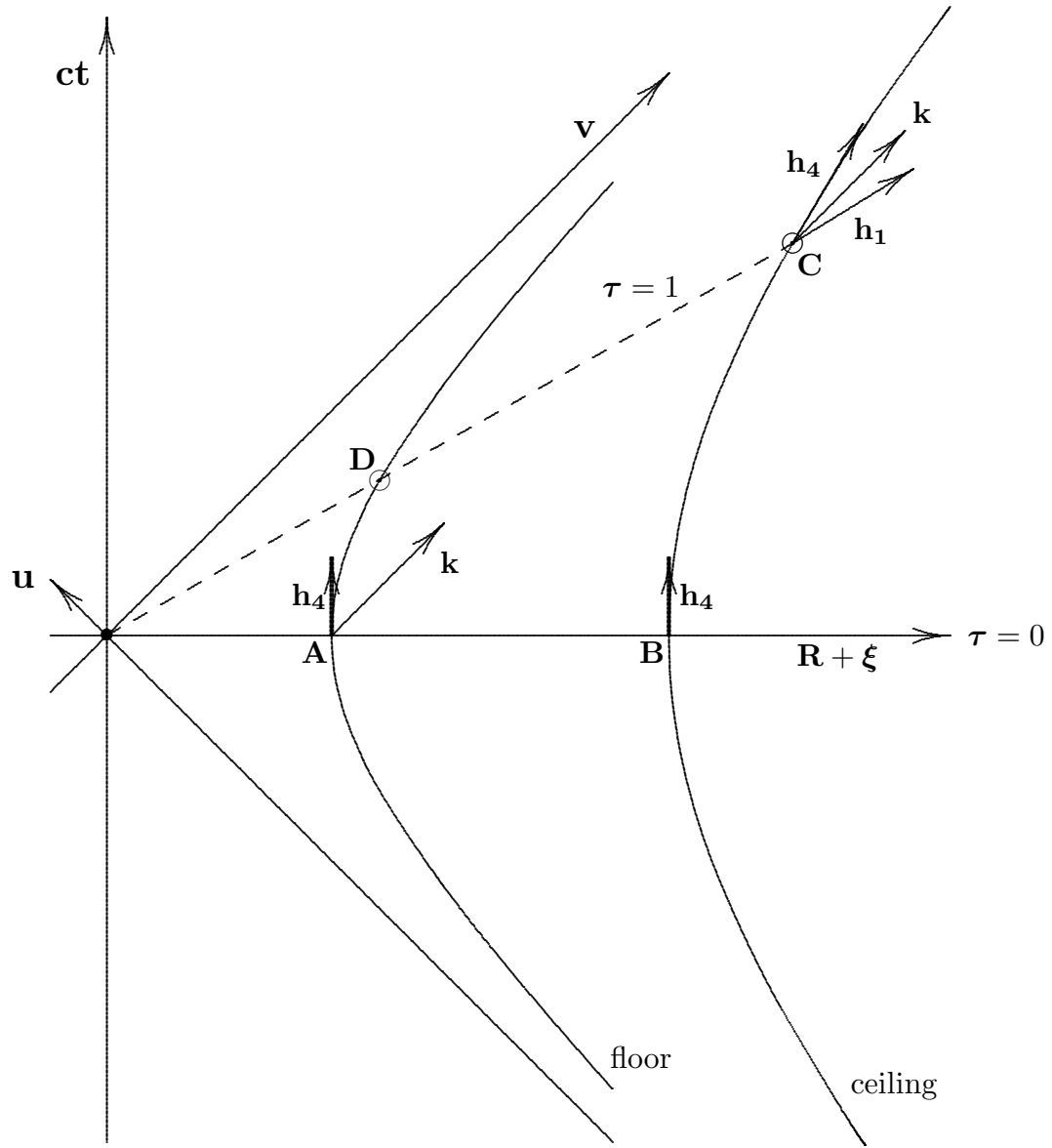
Figure 2: Einstein's system Σ_1, a section through the accelerated Chest in Minkowski space-time. Pseudo-polar coordinates in the Minkowski x-ct-plane underlie a Ricci Grid with unit vectors along its radii and equilateral hyperbolae. The radial direction of increasing ξ is taken as the vertical in Einstein's Chest. Its two horizontal directions are suppressed. The u-direction marks the path of a photon traveling vertically downstairs while the v-direction and the **k**-vector indicate the direction of its world-line for an upwards path of a photon. The equilateral hyperbolic arcs named 'floor' and 'ceiling' are the time-like world lines of two such fixed points in the Chest. They are the analogs of the two circular arcs in Figure 1 and the figure ABCD is its analog rectangle. 'Floor' and 'ceiling' are auto-parallel world-lines describing points at rest in Einstein's accelerated Chest while the radial lines of constant τ mark simultaneous events. ABCD is again a rectangle in Einstein's tele-parallel geometry. A photon emitted from the floor at A will be red-shifted when received at the ceiling since the ceiling has picked up a speed of recession during its travel time. This is indicated by the constancy of the vector **k** in points A and C and the change in the grid vector $\mathbf{h_4}$.
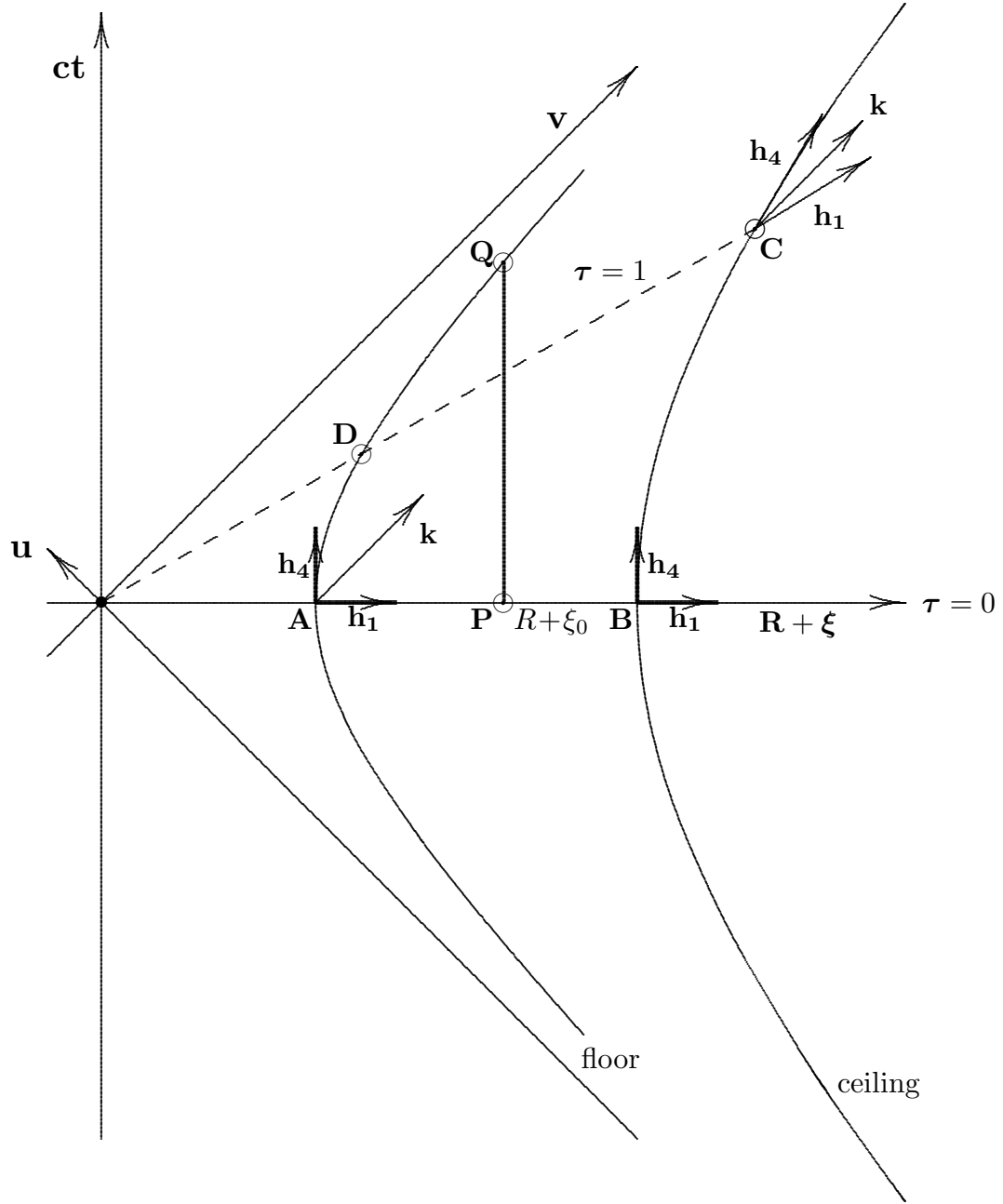
Figure 3: Einstein's system Σ_2, the Chest in a gravitational field
The figure is the same as Figure 2 except that gravitational motion has been intro-
duced through the line PQ. In point P an object resting at time $\tau = 0$ at height ξ_0
above the floor is released. Its world-line is parallel to the time axis ct and will hit
the floor in Q. This motion relative to the Chest is downstairs with acceleration at
point P equal to minus the linear curvature of a hyperbolic arc through P. The fact
that the side BC in the rectangle is longer than the side AD marks the phenomenon

of the gravitational 'redshift': the proper time BC is longer than the proper time AD. That means of two identical clocks, one at the floor and one at the ceiling of Einstein's Chest, the clock at the ceiling goes faster.

Einstein's Apple and Relativity's Gravitational Field
by
Engelbert Levin Schücking

The First Principle of Equivalence

Einstein's Apple

In 1907 Johannes Stark, editor of *The Yearbook of Radioactivity and Electronics,* asked Albert Einstein to write a review of relativity theory for its volume 4. It was an unusual request in Germany's academic world to charge a Second Class Expert at the Swiss Patent Office with the task of a senior professor, to survey recent developments in his field. Einstein's article[1] *On the Relativity Principle and the Conclusions drawn from it* was the result. The last nine pages of this 52 page paper bore the title *Principle of Relativity and Gravitation.* These pages did not review published material; they laid the foundation to Einstein's greatest and most original contribution to science, his theory of gravitation. Fifteen years later he called the epiphany that inspired him *der glücklichste Gedanke meines Lebens*, the happiest thought of my life[2]. In the history of science it is referred to as Einstein's first principle of equivalence and I call it Einstein's Apple. In a speech given in Kyoto, Japan, on December 14, 1922 Einstein remembered his experience[3]:

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[1] Einstein, Albert. "The Collected Papers of Albert Einstein", Volume 2. English Translation. Anna Beck, Translator; Peter Havas, Consultant. Princeton University Press. Princeton NJ 1989. p. 252.

[2] Einstein, Albert.Ref. 1, p. 265.

[3] Pais, Abraham, "Subtle is the Lord". Oxford University Press, New York NY 1982. p. 179.

"I was sitting in a chair in my patent office in Bern. Suddenly a thought struck me: if a man falls freely, he would not feel his weight. I was taken aback. The simple thought experiment made a deep impression on me. It was what led me to the theory of gravity."

This was an unusual vision in 1907. He had not been watching the antics of orbiting astronauts on television, sky-diving clubs did not yet exist, and platform diving was not yet a sport's category of the freshly revived Olympic Games. How could this thought have struck him? Had he just been dealing with patent applications covering the safety of elevators?

A Patent or a Prize as Inspiration?

Three years earlier, in 1904, the Otis Elevator Company installed in Chicago, Illinois, the first gearless traction electric elevator apparatus, that was of the direct drive type, known as the "1:1 elevator". This first modern electric elevator made its way to Europe where, on Zürich's Bahnhof Strasse and elsewhere in Switzerland, buildings went up that needed elevators. It would have been natural for Director Friedrich Haller at the Swiss Patent Office in Bern to put applications involving electromechanical machinery on the desk of Einstein, his expert 2$^{\text{nd}}$ class with expertise in electromagnetism.

We have one patent application with Einstein's comments written just a week after he had finished his last section on Gravitation for Stark's Yearbook. It challenges Germany's electric giant AEG and begins in his neat handwriting[4]:

"The patent claim is incorrect, vague and obscurely redacted."

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[4] Flückinger, Max, "Albert Einstein in Bern". Verlag Paul Haupt Bern 1974. p. 63.

For 7 years, from 1902 to 1909, Einstein reviewed an estimated two-thousand patents. These reviews of patent applications probably constituted the bulk of Einstein's writings in his most productive years. Comparing them with his papers on physics and searching them for clues to his great discoveries might give fascinating insights into the working of his mind. Unfortunately, this attempt at finding a clue to the happiest thought of his life is doomed to failure. The AEG patent application is the only one extant from those years he worked at the patent office. The Swiss bureaucracy did destroy all other examples of Einstein's expert opinions. We shall probably never know how he got the inspiration to his first principle of equivalence.

However, there was one suggestion for the principle of equivalence that went to the heart of the theory. It concerned the apparent enigmatic equality of inertial and passive gravitational mass. In 1906 the Academy of Sciences in Goettingen had offered the Beneke Preis for proving this equality by experiment and theory through an advertisement in the *Physikalische Zeitschrift*. Since this Journal reached practically all German speaking physicists Einstein may have seen the offer. Two months after his Jahrbuch article Einstein published his *Maschinchen* in this Journal.

The Baron Roland Eötvös, the only entry, won three-fourths of this prize (3,400 of 4,500 Marks); only three-fourths, because he had just done experiments and had not attempted a theoretical explanation[5]. It has been claimed that Einstein alone was at that time aware of the importance of the equality of masses.

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[5] Runge, Carl, "Göttinger Nachrichten No.1, p.37-41 (1909).

Einstein's biographer Leopold Infeld wrote:

"No one in our century, with the exception of Einstein, wondered about this law any longer."[6]

Reading Runge's Prize Award one gets the clear idea that the equality of the two masses was at that time the foremost question for theoreticians in Göttingen like Hilbert, Minkowski, Klein, Voigt, Schwarzschild, Runge, Wiechert, and Abraham, when Kaufmann carried out his experiments on the mass of high energy electrons. Comparing the Beneke Prize for 1906 with today's prize money, one may be justified to call the prize for the equality of the masses the $64,000 question. If Einstein thought he had the answer to this question, why did he not compete? I shall come back to this question.

The First Principle of Equivalence

In his 1907 review paper Einstein formulated his principle of equivalence for the first time. He wrote[7]:

"We consider two systems Σ_1 and Σ_2 in motion. Let Σ_1 be accelerated in the direction of its X-axis, and let γ be the (temporally constant) magnitude of that acceleration. Σ_2 shall be at rest, but it shall be located in a homogeneous gravitational field that imparts to all objects an acceleration $-\gamma$ in the direction of the X-axis."

The next sentence contains the principle of equivalence:

"As far as we know, the physical laws with respect to Σ_1 do not differ from those with respect to Σ_2; this is based on the fact that all bodies are equally accelerated in the gravitational

[6] Infeld, Leopold, "Albert Einstein", Charles Scribner's Sons , New York 1950, p. 47.
[7] Einstein, Albert. Ref.1, p. 302.

field."

[It was this last fact that had prompted Sir Hermann Bondi to the observation "If a bird-watching physicist falls off a cliff, he doesn't worry about his binoculars, they fall with him."]

Einstein continued:

"At our present state of experience we have thus no reason to assume that the systems Σ_1 and Σ_2 differ from each other in any respect, and in the discussion that follows, we shall therefore assume the complete physical equivalence of a gravitational field and a corresponding acceleration of the reference system."

The Role of Special Relativity.

So far I did discuss Einstein's first principle of equivalence in terms of Newton's theory of gravitation. Einstein had set himself the task of studying how the principles of special relativity from his 1905 paper[8] *On the Electrodynamics of Moving Bodies* would affect Newton's theory of gravitation. Looking back from a century later, we say that special relativity is based on the representations of the Poincaré group of space-time translations and Lorentz transformations while Newton's theory was based on those of the Galilei group. This means that Newton's theory was simply incompatible with special relativity.

Einstein's principle of equivalence gave him the clue to search for a theory of gravitation based on special relativity. If gravitation was locally nothing but a description of space and time from an accelerated reference frame, he could succeed by studying accelerated reference frames in special relativity. And this is

[8] Einstein, Albert. Ref. 1, p. 140.

what he did. His deep physical intuition led him to two crucial conclusions:

Two identical clocks at rest in a gravitational field will show a relative difference $\Delta\nu/\nu$ in their rate ν given by $\Delta\Phi/c^2$, their difference $\Delta\Phi$ in gravitational potential Φ divided by the square of the velocity c for light in a vacuum. The clock on the higher potential, e.g., in the earth field at higher elevation, would run slightly faster. This was first demonstrated in a terrestrial experiment[9] by Robert Pound and Glen Rebka in 1960, five years after Einstein's death. He had not foreseen an experiment on earth as a test where for a difference in height of ten meters $\Delta\nu/\nu$ is 10^{-15}, a millionth of a billionth. Einstein wrote[10]

"There exist 'clocks' that are present at locations of different gravitational potentials and whose rates can be controlled with great precision; these are the producers of spectral lines. It can be concluded from the aforesaid that the wavelength of light coming from the sun's surface, which originates from such a producer is larger by about one part in two million than that of light produced by the same substance on earth."

It was only after 1960, as the conditions on the solar surface were better understood, that Einstein's prediction for the sun could be confirmed.

The other important result of his investigation concerned the validity of his formula $E = mc^2$. He stated at the end of his article[11]:

"Thus to each energy E in the gravitational field there cor-

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[9] Pound, R. V. and Rebka, G.A. "Apparent Weight of Photons", Phys. Rev. Lett. **4**, 337-341 (1960).

[10] Einstein, Albert. Ref. 1, p. 307. Einstein's solar redshift is 2×10^{-6}, not, as translated here, "one part in two millionth".

[11] Einstein, Albert. Ref. 1, p. 311.

responds an energy of position that equals the potential energy of a 'ponderable' mass of magnitude E/c^2. Thus the proposition, that to an amount of energy E there corresponds a mass of magnitude E/c^2, holds not only for the *inertial* but also for the *gravitational* mass, if the assumption introduced in Section 17 [the first principle of equivalence] is correct."

Einstein also pointed out that in a gravitational field light rays, not in the direction of the acceleration, would be bent. But the correct formula for this process was not yet attained.

A Forgotten Berichtigung

Since explanations of Einstein's equivalence principle are usually given without the use of special relativity, a crucial detail of its formulation remains often unmentioned. In Newtonian Theory this principle is true for extended bodies moving with arbitrary velocities since the gravitational acceleration in this theory is independent of velocity. In Relativity Theory this is no longer the case. Here acceleration of a particle is a vector that is always orthogonal to the tangent 4-vector of its world line. The notion of *relative* acceleration exists only for particles whose four-velocities agree. Interpreting the gravitational acceleration of a falling object as minus the acceleration of the reference system had to be restricted to objects at rest. When Einstein wrote his Jahrbuch article in 1907 he was apparently not aware of this limitation. It was a letter from Max Planck that had alerted him to this fact when he published a *Berichtigung* (an Erratum) in the 1908 Jahrbuch[12].

The two systems Σ_1 and Σ_2 of his Jahrbuch article, one ac-

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[12] Einstein, Albert Ref. 1. p. 317.

celerated and the other in a gravitational field, must, therefore, not be considered in motion with respect to each other. One has to think of the two systems as one and the same system. There is no Poincaré transformation between Σ_1 and Σ_2 different from the identity. What distinguishes the two systems is their different dynamical interpretation: the question whether Einstein's system Σ is accelerated or suspended in a gravitational field. There are also relativistic problems for extended bodies since a homogeneous gravitational field in Minkowski's space-time modifies the translation group. Relativity theory thus restricts the validity of the Newtonian principle of equivalence to a local space-time event or to a single world line. This was acknowledged by Einstein in the *Erratum*. His considerations could only work for small velocities, for small accelerations, only in a small neighborhood of an event in space. He did not have the necessary concepts and mathematical theories available to discover the relativistic gravitational field. Even Minkowski's beautiful space-time picture that would have been of help was demonstrated only later that year. The recognition that his brilliant idea did not point the way to an extended relativistic gravitational field must have been devastating. It would have discouraged him to apply for the Beneke Prize if he had ever considered it.

Ten years after his Yearbook article Einstein described his first equivalence principle in loving detail in an account of the new theory of gravitation in his book[13] *Relativity, the Special and the General Theory.* This description became the archetype of the Einstein elevator. The elevator car, called

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[13] Einstein, Albert. "Relativity". Penguin Books. NYC, NY 2008. p. 63.

Kasten by Einstein, was Englished into a *chest* by his authorized translator Robert W. Lawson, a British physicist who had studied German as a prisoner of war in Austria. Einstein allowed his chest be pulled with constant acceleration to reach arbitrary high velocities without mentioning the limitations of Special Relativity. He apparently tried to erase the *Berichtigung* from his memory. As far as I know, Einstein never referred to his *Erratum* again. Nor do his biographers.

The Relativistic Gravitational Field

Twenty Years Later

In May 1928 Einstein was bedfast with pericarditis. He wrote to his friend Heinrich Zangger in Zürich[14]: "In the tranquility of illness I have laid a wonderful egg in the domain of General Relativity. Whether the bird hatching from it will be vigorous and long-lived lies on the knees of the gods. Meanwhile I approve the illness that so has blessed me." On July 10, 1928 the bird appeared as *Riemann-geometry with keeping the Notion of Tele-parallelism.* In the introduction to his paper[15] he explained:

"Characteristic for Riemann's geometry is that the infinitesimal neighborhood of every point P has a Euclidean metric and that the length of two line-elements belonging to the infinitesimal neighborhoods of two points P and Q at a finite distance

[14] Einstein, Albert. "Einstein Archives", The Hebrew University of Jerusalem (EA), call no. 40–69. Reproduced in "Albrecht Fölsing, *Albert Einstein*. Suhrkamp Verlag Frankfurt am Main 1994. Third Edition. p. 684.

[15] Einstein, A. "Riemann-Geometrie mit Aufrecterhaltung des Begriffes des Fernparallelismus". *Sitzungsberichte der Preußischen Akademie der Wissenschaften Phys.-Math. Klasse. 1928.XVII.* p. 217.

from each other can be compared to each other. However, the notion of parallelism of two such line-elements is missing. The notion of direction does not exist for the finite. The theory to be proposed in the following is characterized by the fact that it introduces for the finite besides the Riemannian metric the 'direction', or rather the equality of direction, or the parallelism."

Ricci's Grid

In his paper Einstein defined parallelism by using Ricci's *ennuples* that he now named *n-Beine* (*n*-legs). He called two vectors parallel if their frame components were proportional. In a Euclidean space with frames based on ortho-normal Cartesian coordinates this would result in Euclid's definition of parallelism. However, based on Ricci's ennuples it extended the notion. In particular, this meant now that parallel vectors of the same length had identical frame components everywhere.

It was the great Italian geometer Gregorio Ricci-Curbastro who generalized the concept of 3-dimensional frames $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in the Euclidean space to such ortho-normal frames in n-dimensional Riemannian manifolds. He introduced them in his 1895 paper[16] *On the Theory of Hyperspaces* and called them later "ennuples". Ricci's ennuples formed a frame of n unit vectors $\mathbf{h}_j$ with $j = 1, \ldots, n$. An arbitrary vector field $\mathbf{a}(x^\lambda)$ at a point with coordinates x^λ would then be represented by its frame components $a_j(x^\lambda)$ in terms of the unit vectors $\mathbf{h}_j(x^\lambda)$ as

$$\mathbf{a} = a_j \, \mathbf{h}_j \equiv \Sigma_{j=1\ldots n} \, a_j \, \mathbf{h}_j \, . \tag{1}$$

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[16] Ricci-Curbastro, Gregorio, "Sulla Teoria degli Iperspazi" *Rend. Acc. Lincei Serie IV* (1895). p. 232–237. Reprinted in : Gregorio Ricci-Curbastro, Opere, vol. I. Edizioni Cremonense Roma 1956. p. 431–437.

The $\mathbf{h}_j$ were subject to the ortho-normality conditions

$$\mathbf{h}_j \cdot \mathbf{h}_k = \delta_{jk}, \quad j, k = 1, 2, \ldots, n \qquad (2)$$

with constants δ_{jk} vanishing for $j \neq k$ and equal to one for $j = k$.

Vector fields can be visualized as stream-lines of a stationary flow or as Faraday's lines of force. A non-vanishing vector field in space generates a space-filling system of lines through each point, known to mathematicians as a *congruence*. Ricci's vision[17] filled Riemann's n-dimensional space with n congruences orthogonal to each other creating a framework for the physical components of vectors and tensors. This scaffold is not tied to the coordinates. It serves as a reference body for measurements in Riemann's manifold with Ricci's ennuples as tangent unit vectors in each point providing the local scale along each line. I call this reference body "Ricci's Grid".

Einstein Discovers Ricci's Grid

In 1901 Ricci published, together with his student Tullio Levi-Civita, a review paper[18] of his ingenious system of n-dimensional tensor analysis with its clever use of upper and lower indices. This paper, written in French by two Italians for a German journal, became the source from which Einstein, helped by Marcel Großmann, derived the formal tools for his theory of gravitation. Einstein did not find it easy learning the new formalism.

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[17] Ricci-Curbastro, Gregorio "Dei sistemi di congruenze ortogonali i una varietà qualunque" *Mem. Acc. Lincei Serie 5 vol II* (1896). p. 276–322. Reprinted in: Gregorio Ricci-Curbastro Opere Vol II Editore Cremonense. Roma 1957. p. 1–61

[18] Ricci, Gregorio, and Levi-Civita, Tullio. "Méthodes de calcul différentielle absolu et leurs applications." *Mathematische Annalen* 54 (1901) p. 125–201.

His friend Louis Kollros[19] remembered from that time Einstein's scream "Großmann, you've got to help me or I'm going nuts!"

Einstein wrote to Arnold Sommerfeld[20]:

"I am now working exclusively on the gravitation problem and believe that I can overcome all difficulties with a mathematical friend of mine here. But one thing is certain: never before in my life have I troubled myself over anything so much, and I have gained enormous respect for mathematics, whose more subtle parts I considered until now, in my ignorance, as pure luxury! Compared with this problem, the original theory of relativity is child's play".

At first, Einstein did not use the full advantage of the formalism and seems to have confined his study to the first chapter of the tensor bible that did not use Ricci's ennuples and always worked with coordinate components of vectors and tensors. It was apparently only in 1928 when he discovered his new geometry based on Ricci's ennuples that Einstein apparently had progressed to the second chapter bearing the title *La Géométrie intrinseque comme instrument de calcul.*

Before discussing Einstein's new geometry it will be useful to look at two examples, first its simplest derived from polar coordinates in the Euclidean plane.

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[19] Kollros, L. "Albert Einstein en Suisse Souvenirs" in *Fünfzig Jahre Relativitätstheorie* Helvetica Physica Acta Supplementum IV 1956.

[20] Einstein, Albert. "The Collected Papers of Albert Einstein" volume 5. English Translation. Anna Beck, Translator. Don Howard, Consultant. Princeton UP Princeton NJ 1995. p.324.

Plane Polar Coordinates

The metric of the Euclidean plane written in polar coordinates r and θ is given by

$$ds^2 = dr^2 + r^2 \, d\theta^2 \, . \tag{3}$$

Since the lines $r = const$ and $\theta = const$ intersect under right angles the frame of their unit tangent vectors $\mathbf{h}_1$ and $\mathbf{h}_2$ can be chosen as:

$$\mathbf{h}_1 = \partial/\partial \mathbf{r} \, , \quad \mathbf{h}_2 = r^{-1} \partial/\partial \theta \, . \tag{4}$$

In Einstein's geometry the auto-parallel lines are those whose tangent vector is always parallel to itself. Those are the *straightest* lines in his geometry. For the frames adapted to plane polar coordinates such lines will intersect the lines $\theta = const$ under a fixed constant angle. Their equation is given by

$$r = a \, e^{\beta \theta}, \quad a > 0 \, , \quad \beta \neq 0 \, , \tag{5}$$

with real constants a and β. They are known as logarithmic spirals. The coordinate lines themselves are their degenerate cases

A remarkable feature for rectangles of straightest lines in Einstein's geometry is the following: a rectangle formed by the two auto-parallels $r = r_1$ and $r = r_2$ and the other pair of auto-parallels $\theta = \theta_1$ and $\theta = \theta_2$ will have one set of opposite sides with equal length $r_2 - r_1$ and the other set with unequal lengths $r_2(\theta_2 - \theta_1)$ and $r_1(\theta_2 - \theta_1)$, respectively (Fig. 1).

Einstein's Chest in Minkowski's Space-Time

Einstein's chest was accelerated in the direction of the x-axis with constant acceleration γ. Such a motion was discussed by

Max Born[21] in 1909. For the description of this system I can use Ricci's ennuples in Minkowski space-time with metric

$$ds^2 = dx^2 + dy^2 + dz^2 + (icdt)^2 \,. \tag{6}$$

A point on the floor of the chest describes, due to its constant acceleration, a world line that is an equilateral hyperbola with constant R.

$$(x/R)^2 + (ict/R)^2 = 1 \,, \quad x > 0 \,. \tag{7}$$

The acceleration, the inverse radius of this pseudo-circle, is $\gamma = c^2/R$ where c is the speed of light. For $\gamma = 9.8 \, ms^{-2}$, R equals one light-year. If h is the height of the chest, the equation for a point on its ceiling is

$$\left(\frac{x}{R+h}\right)^2 + \left(\frac{ict}{R+h}\right)^2 = 1 \,, \quad x > 0 \,. \tag{8}$$

Its acceleration, given by $\gamma = c^2/(R+h)$, is just a tiny bit smaller. For the gravitational picture it is convenient to introduce coordinates in the x-ct-plane of Minkowski's space-time by the analogous transition from Cartesian to polar coordinates

$$x = (R+\xi)\,Ch\,\tau \,, \quad ct = (R+\xi)\,Sh\,\tau \,. \tag{9}$$

The Minkowski metric takes then the form

$$ds^2 = d\xi^2 + dy^2 + dz^2 - (R+\xi)^2\,d\tau^2 \,. \tag{10}$$

While the hyperbolae $\xi = const$ in the ξ-τ-plane are the analogs of concentric circles in the Euclidean polar coordinates, the lines of equal time τ correspond to the radii of those circles. This

[21] Born, M. Annalen der Physik, Leipzig. Volume 30 (1909). p. 1.

coordinate system is orthogonal in the space-time metric. In the new coordinates the frame vectors are drawn along the coordinate lines. The frame vectors $\mathbf{h}_j$ are in the chest

$$\mathbf{h}_1 = \partial/\partial\xi, \quad \mathbf{h}_2 = \partial/\partial y,$$
$$\mathbf{h}_3 = \partial/\partial z, \quad \mathbf{h}_4 = i(R+\xi)^{-1}\partial/\partial\tau. \qquad (11)$$

The fourth vector, tangent to the hyperbolae, is an imaginary one. The acceleration in the chest at a height ξ above the floor is given by $\gamma = c^2/(R + \xi)$ The radius of curvature of the hyperbolic world line of any fixed point in the chest shows the state of acceleration in Einstein's system Σ_1 (Fig. 2).

The Chest Viewed as a Gravitational Field

However, I can also view the chest as system Σ_2 by using Einstein's geometry. Then fixed points in the chest proceed on auto-parallel world lines with tangent unit 4-vector $\mathbf{h}_4$ defining a state of rest in the chest. If an object is dropped in the chest it will, neglecting air resistance, describe a geodesic world line. If the object is released at time $\tau = 0$ at height ξ_0 above the floor its motion is described by the equations

$$R + \xi_0 = (R+\xi)\,Ch\,\tau, \quad ct = (R+\xi)\,Sh\,\tau, \qquad (12)$$

with $(y, z) = const$. This gives for small times t

$$\xi = \xi_0 - \frac{1}{2}\frac{c^2}{R+\xi}t^2, \qquad (13)$$

leading, at $t = 0$, to an acceleration $\gamma = -c^2/(R + \xi_0)$. The object dropped at $t = 0$ proceeds on the geodesic $x = R + \xi_0 = const$. At $t = 0$ it's 4-velocity $\mathbf{u}$, tangent to the geodesic, is

equal to the frame vector $\mathbf{h}_4$. At that moment its gravitational acceleration equals minus the acceleration of the system Σ_1. The gravitational field is not exactly homogeneous. There exist homogeneous gravitational fields in Minkowski space-time with constant acceleration[22]. I have chosen here the case of a static field because it is described by the *Rindler coordinates* that go back to Levi-Civita[23].

The Gravitational Red-shift

If a light wave is emitted vertically upwards from the floor of Einstein's chest, its frequency ν will be red-shifted when received at the ceiling of the chest. It is sometimes claimed that the existence of such a gravitational red-shift forces us to admit a non-vanishing space-time curvature[24]. These arguments are not convincing since we are dealing with flat Minkowski space-time. What the arguments actually get at is the Einstein geometry of Minkowski space-time when interpreted as a static gravitational field.

The existence of the red-shift is immediately clear in the acceleration picture Σ_1 where source and receiver of the light wave are accelerated. If the wave is emitted upwards from the floor, then by the time of reception the receiver at the ceiling will have acquired a speed V of recession of about $V = \gamma h/c$ if $V << c$.

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[22] Schucking, Engelbert and Surowitz, Eugene. "Einstein Fields". A book manuscript, unpublished.

[23] Levi-Civita, Tullio, "Statica Einsteiniana," Rendiconti della R. Accademia dei Lincei, ser. 5, 1st sem., vol. 26 (1917). p. 458.

[24] Schild, Alfred. Lectures in Applied Mathematics, Vol. 8. American Mathematical Society, Providence R.I. (1967).

The red-shift appears then as the Doppler shift

$$-\frac{\delta\nu}{\nu} = \frac{V}{c} = \gamma\frac{h}{c^2} = \frac{\Delta\lambda}{\lambda}. \qquad (14)$$

The gravitational red-shift has a simple explanation in Einstein's geometry (Fig. 3). One can refer to the discussion of polar coordinates on the Euclidean plane. Drawing in the ξ-τ-plane the rectangle bounded on bottom and top by the auto-parallel lines $\tau = 0$ and $\tau = 1$, respectively, and adding its left and right sides by the auto-parallel lines $\xi = 0$ and $\xi = h$, respectively, one notices that opposite sides of the rectangle are parallel. Top and bottom of the rectangle have both length h. The left side of the rectangle has *length R*, i.e., proper time R/c, while the right side has *length $R + h$*, i.e., proper time $(R + h)/c$. This missing length h, or missing proper time h/c, is the geometrical description of the gravitational red-shift in Einstein's geometry. The relative gap h/R equals the relative change in wavelength $\Delta\lambda/\lambda$. It says in physical terms that a standard clock at the ceiling of Einstein's chest runs faster than a standard clock on the floor. Such non-closure of rectangles, or parallelograms, is the typical feature of Einstein's geometry. It has nothing to do with the curvature of space-time since the Minkowski plane has zero curvature.

The analogy with the Einstein geometry in the Euclidean plane based on polar coordinates goes even further than expected. If one asks for the auto-parallel lines of constant velocity β in the vertical direction one is looking for the analogs of the logarithmic spirals in the Euclidean plane. Their equation

turns out to be the same, except for new names of the variables

$$R + \xi = a\,e^{\beta\tau}\,. \tag{15}$$

Jakob Bernoulli was the first using polar coordinates systematically and studied the properties of the logarithmic spiral. He became so fond of it he willed this curve engraved on his tombstone at the Münster in Basel, Switzerland, where it can still be seen today. It carries the inscription *eadem mutata resurgo* (Though changed I rise again). Unfortunately, the stone mason made it an Archimedean spiral.

What Is a Gravitational Field?

The brief answer is: A GRAVITATIONAL FIELD IS A TELE-PARALLEL RICCI GRID.

Its mathematical characterization is quite simple: the commutator of the tele-parallel vector fields of the Grid

$$[\,\mathbf{h}_j,\,\mathbf{h}_k\,] = -\,\mathbf{h}_l\,T^l_{\ jk}\,, \quad T^l_{\ jk} = -\,T^l_{\ kj}\,, \tag{16}$$

gives rise to the torsion tensor $T^l_{\ jk}$ that is skew-symmetric in its lower indices.

(If we use an imaginary time-like vector we need not distinguish between upper and lower indices.) This three-index tensor describes the field strength of the gravitational field. For the example in equation (11) we obtain for the only component of the torsion tensor different from zero

$$T^4_{\ 14} = -\,T^4_{\ 41} = 1/(R+\xi)\,. \tag{17}$$

One must wonder why Einstein did not recognize that his new geometry of the tele-parallel Ricci Grid ended the long-sought

search for the tensorial description of his relativistic gravitational field. Here was the precise definition of the *reference body* necessary for measurements and the elusive reference mollusk[25]. The Ricci Grid was just a mathematical model for the matter serving as measuring instrument for the field.

I can only speculate why this wasn't obvious when he introduced his new geometry for Riemannian manifolds:

Since the beginning of the 1920's Einstein had largely dropped research on his theory of gravitation though he still supervised work of collaborators on, e.g., the theory of motion in General Relativity. Instead, Einstein had become obsessed with finding a home for electromagnetism in a geometrical theory that included gravitation. This question occupied him for the last third of his life[26]. When he discovered *Fernparallelismus*[27] in 1928 he wanted to use its 4-dimensional geometry for a field theory combining electromagnetism with gravitation. However, he saw in the frame vectors not the description of the reference body but the manifestations of E&M. Einstein believed that the tensor T^l_{jk} contained besides the degrees of freedom of the gravitational field also those of electromagnetism, e.g., initially he identified a contraction of the tensor with a multiple of the electromagnetic four-potential. It was not clear to him that the tensor T^l_{jk} was simply equivalent to the tensor of accelerations.

[25] Ref. 13, p. 112.
[26] Goenner, Hubert "On the History of Unified Field Theories" *Living Reviews in Relativity* http://www.livingreviews.org/lrr-2004-2
[27] Sauer, Tilman "Field equations in teleparallel spacetime: Einstein's *Fernparallelismus* approach towards unified field theory". arXiv:physics/0405142v1 26 May 2004 and HISTORIA MATHEMATICA [doi101016/jhm2005.11.005] .

The First Principle of Equivalence, Final Version

The Susskind Principle of Equivalence

"THE EQUIVALENCE PRINCIPLE: GRAVITY IS INDISTINGUISHABLE FROM ACCELERATION."[28]

This brief formulation expressed, perhaps, Einstein's aim to generalize his first principle of equivalence that he had formulated for homogeneous gravitational fields in 1907. Since we now have a precise invariant definition of a gravitational field, we can investigate whether Susskind's formulation can be justified. For that purpose we need a definition of acceleration. It is sufficient to have such an expression for the basis vectors $\mathbf{h}_j$. It is given by Ricci's covariant derivation of the vector $\mathbf{h}_j$ in direction $\mathbf{h}_k$ written as

$$\nabla_{\mathbf{h}_k}\mathbf{h}_j \;=\; -\,\gamma^l{}_{jk}\,\mathbf{h}_l\,, \tag{18}$$

where the γ_{jkl} are known as Ricci's rotation coefficients. In his brief 1895 paper "Sulla Teoria Degli Iperspazi" that introduced the coefficients, Ricci showed that they were skew-symmetric in their first pair of indices[29]

$$\gamma_{jkl} \;=\; -\,\gamma_{kjl}\,. \tag{19}$$

Ricci's covariant derivative has vanishing torsion. That is expressed by the equation

$$\nabla_{\mathbf{h}_k}\mathbf{h}_j \;-\; \nabla_{\mathbf{h}_j}\mathbf{h}_k \;-\; [\,\mathbf{h}_k,\,\mathbf{h}_j\,] \;=\; 0\,. \tag{20}$$

Using equation (16) gives with equations (18) and (20) the

[28] Susskind, Leonard. "The Cosmic Landscape". Little, Brown and Co. New York, NY 2005 p. 347.
[29] Ricci-Curbastro, Gregorio Ref. 17.

FINAL VERSION OF THE EQUIVALENCE PRICIPLE:

$$- \gamma^l{}_{jk} + \gamma^l{}_{kj} = T^l{}_{jk} \,. \tag{21}$$

The left-hand side of this equation is given by the acceleration of the Ricci Grid, while the right-hand side carries the tensor of the gravitational field strength. The right-hand side determines also the Ricci rotation coefficients that form the contorsion tensor. Using equation (19) one easily derives from (21) that

$$\gamma_{ljk} = \frac{1}{2} \left(- T_{ljk} + T_{klj} - T_{jkl} \right) \,. \tag{22}$$

This equation confirms that we have a true equivalence of gravitation and acceleration.

Ricci's Coefficients in Space-Time

A Ricci Grid at an event in space-time is a mathematical model for an infinitesimal rigid body at rest defining units of length and time. A different ennuple at the same event is obtained through a Lorentz transformation of the ennuple. The six parameters of that transformation can be given by an angular orientation vector $\boldsymbol{\phi}$ and a velocity vector $\mathbf{v}$. A neighboring ennuple differs by an infinitesimal Lorentz transformation described by the skew-symmetric first pair of indices in Ricci's γ-tensor. The third index of the tensor gives the gradient of the infinitesimal Lorentz transformation.

If indices a, b, c run through the numbers from 1 to 3, $- i\, \gamma_{ab4}$ describe angular velocity and $- \gamma_{4a4}$ acceleration against absolute space for the infinitesimal rigid body.

Further, γ_{abc} are the spatial gradients of the orientation vector $\boldsymbol{\phi}$ for the rigid body, while $- i\, \gamma_{4ab}$ is the spatial gradient

of the vector **v**. Altogether, one can say that the coefficients are a generalization of the Pauli-Lubanski vector for the vectors in Ricci's ennuples. Through the split into space and time one describes all 24 components of what we call *acceleration*.

Some History of the Tele-Parallel Ricci Grid

Hessenberg

When Einstein discovered tele-parallelism in 1928 he was apparently not aware of the fact that the tele-parallel Ricci Grid had been discovered a dozen years earlier. The man who first recognized this possibility was a professor of mathematics in Breslau, well known in the profession for his work on the foundations of geometry. In a paper finished in June 1916 Gerhard Hessenberg[30] replaced Christoffel's cumbersome calculations by an invariant co-vector method that reached later an even more elegant form in Élie Cartan's papers[31]. Hessenberg discovered the torsion tensor T_{kjl} and the contortion tensor. By introducing auto-parallel curves for a tele-parallel Ricci Grid he proved that the torsion tensor vanishes if and only if all auto-parallel curves are geodesics, that is, shortest lines in the Riemann metric. Hessenberg was the first to discover that the geometry of a gravitational field is characterized by the torsion of teleparallelism.

I have nowhere seen his discovery of this special kind of torsion acknowledged. Hermann Weyl[32], who missed finding

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[30] Hessenberg, Gerhard, "Vektorielle Begründung der Differentialgeometrie". Mathematische Annalen vol. 78, p. 187–217 (1917).

[31] Cartan, É. "Riemannian Geometry in an Orthogonal Frame". World Scientific. Singapore 2001.

[32] Weyl, H. "Raum . Zeit . Materie". Springer Verlag 7$^{\text{th}}$ ed. 1988. p.332.

torsion when generalizing the notion of connections, refers to Hessenberg only by crediting him with the proof that the symmetry of the Riemann tensor in its first and second pairs of the indices follows from the cyclic symmetry in the last three indices. Appendix A offers a few notes about Hessenberg's 1916 paper.

Cartan

Torsion was named by Élie Cartan[33] and announced in a 3 page note in Comptes Rendus in March 1922 with the title *A Generalization of the Riemann Curvature and the Spaces with Torsion*. Einstein learned about this new revolutionary concept in geometry only four weeks later. His friend Paul Langevin had invited him to lecture at the Collège de France in Paris on March 31, 1922.

In the aftermath of World War I, the first lecture by a professor from the country of the archenemy was a highly charged political affair. To cut down on demonstrations, it was by invitation only, and the French Prime Minister Paul Painlevé stood at the door checking. During this lecture week, Jacques Hadamard, professor at the Collège de France gave a party for Einstein. Among his guests, who was to meet Einstein there, was Élie Cartan, the world authority on Lie algebras and the greatest geometer of his time. Cartan thought torsion might have important physical applications and used the occasion to tell Einstein about his recent discovery. He tried to explain the novel concept to him by the example well known to map

[33] Cartan, É. Comptes Rendus Acad. Sci.1922. vol. 174, p. 593.

makers and navigators of tele-parallelism arising from polar coordinates on the sphere. But apparently neither Cartan nor Einstein realized that this example held the key to Einstein's first equivalence principle. Einstein[34] wrote to Cartan 7 years later about his introduction to tele-parallelism:

"I didn't at all understand the explanations you gave me in Paris; still less was it clear to me how they might be made useful for physical theory."

Weitzenböck

The occasion of his letter[35] was that Einstein had attempted using tele-parallelism for a generalized field theory of gravity and electromagnetism and Cartan had reminded him that tele-parallelism was a special case of Cartan's torsion. But since Einstein's first paper on torsion in 1928 he had learned that Roland Weitzenböck had also published papers on torsion. In fact, in his paper "Differential Invariants in Einstein's Theory of Tele-parallelism" Weitzenböck[36] had given a supposedly complete bibliography of papers on torsion without mentioning Elie Cartan. A bizarre circumstance suggested that this omission may have been deliberate. Weitzenböck of Amsterdam University in the Netherlands was an Austrian K. u. K. army officer before WWI. In 1923 he had published a modern monograph on the Theory of Invariants that included Tensor Calculus.[37] In the innocent looking Preface, one finds that the first letter of

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[34] Debever, Robert (ed.) "Élie Cartan – Albert Einstein. Letters on Absolute Parallelism 1929–1932". Princeton University Press. 1979.

[35].Debever, Ref. 51. p.4

[36] Weitzenböck, R., Sitzungsberichte der Preußischen Akademie der Wissenschaften phy.-math. Klasse (Berlin) 1928. p. 466.

[37] Weitzenböck, R. "Invariantentheorie". Nordhoff, Groningen 1923.

the first word in the first 21 sentences spell out:

NIEDER MIT DEN FRANZOSEN (down with the French).

To set the record straight, Einstein invited Cartan[38] to write about the history of torsion in the Annalen der Mathematik. Neither Cartan, nor Weitzenböck in their history of torsion mentioned Hessenberg who had died in 1925.

What is General Relativity?

According to Synge and Bondi

When after Einstein's death in 1955 a new generation of differential geometers and mathematically oriented theoretical physicists re-discovered General Relativity, the vague statements on equality of all motions based on the first principle of equivalence had lost their appeal. The Irish mathematician John Synge wrote in 1960 in the Introduction to his book[39] *Relativity, the General Theory*: "...I have never been able to understand this principle." And he went on: "Does this mean that the effects of a gravitational field are indistinguishable from the effects of an observer's acceleration? If so, it is false. In Einstein's theory, either there is a gravitational field or there is none, according as the Riemann tensor does or does not vanish. This is an absolute property; it has nothing to do with any observer's worldline. Space-time is either flat or curved, and in several places of the book I have been at considerable pains to separate truly gravitational effects due to curvature of space-time from those due to curvature of the observer's worldline

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[38] Cartan, É. Annalen der Mathematik, vol.102 (1930) p. 698.

[39] Synge, John, "Relativity: The General Theory. Elsevier. New York, NY 1960. Preface, p.IX–X.

(in most ordinary cases the latter predominate). The Principle of Equivalence performed the essential office of midwife at the birth of general relativity, but, as Einstein remarked, the infant would never have got beyond its long-clothes had it not been for Minkowski's concept. I suggest that the midwife be now buried with appropriate honours and the facts of absolute spacetime be faced."

At Einstein's Centenary in 1979 Hermann Bondi celebrated him with the essay [40] *Is "General Relativity" Necessary for Einstein's Theory of Gravitation?* Bondi wrote: "From this point of view, Einstein's elevators have nothing to do with gravitation, they simply analyse inertia in a perfectly Newtonian way. Thus the notion of general relativity does not in fact introduce any post-Newtonian physics; it simply deals with coordinate transformations. Such a formalism may have some convenience, but physically it is wholly irrelevant. It is perhaps rather late to change the name of Einstein's theory of gravitation, but general relativity is a physically meaningless phrase that can only be viewed as a historical memento of a curious philosophical observation."

In the unsuccessful attempt of finding a mathematical formulation for General Relativity the field of research narrowed into "Einstein's Theory of Gravitation". But Einstein's Theory of Gravitation was, simply speaking, "Einstein's Field Equations". What then was General Relativity built on the principle of equivalence?

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[40] Bondi, Hermann in "Relativity, Quanta, and Cosmology" edited by Francesco de Finis. Volume1. Johnson Reprint Corporation. New York, NY 1979. p.181.

Einstein Vindicatus

A gravitational field defined as a tele-parallel Ricci Grid is, in general, a non-inertial system for a finite region of the space-time manifold. It also defines a reference body in a state of acceleration and rotation against absolute space-time. However, such gravitational fields appear already in Minkowski space-time.

We consider now generalized Lorentz transformations of the Ricci Grid. By that we mean space-time dependent transformations

$$\mathbf{h}'_m = A_m{}^j(x^\lambda)\,\mathbf{h}_j\,, \quad A_{mj}\,A^m{}_k = \delta_{jk}\,. \qquad (23)$$

Such an infinitesimal transformation is given by

$$A_m{}^j = \delta_m{}^j + \varepsilon_m{}^j\,, \quad \varepsilon_{mk} = -\,\varepsilon_{km}\,, \qquad (24)$$

where the ε_{mk} are infinitesimal skew-symmetric functions of the coordinates. Besides the tensor transformation of the gravitational field strength $T^l{}_{jk}$ the terms

$$\delta T^l{}_{jk} = d\,\varepsilon_k{}^l\,(\mathbf{h}_j) - d\,\varepsilon_k{}^l\,(\mathbf{h}_k) \qquad (25)$$

now appear due to the space-time dependence of the Lorentz transformations. GENERAL RELATIVITY MEANS: INVARIANCE UNDER GENERALIZED LORENTZ TRANSFORMATIONS.

As the Korean physicist Y.M. Cho showed in his paper[41] *Einstein Lagrangian as the translational Yang-Mills Lagrangian*

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[41] Cho, Y. M., Physical Review D 14 (1976), 2521.

that the Lagrangean density

$$\pounds = \kappa^{-2} \left(- \det h^{\mu}{}_j \right)^{1/2} \left[\frac{1}{4} T_{ijk} \, T_{ijk} + \frac{1}{2} T_{ijk} \, T_{ijk} - T_{ijj} \, T_{ikk} \right] \tag{26}$$

giving the Einstein vacuum field equations is, in fact, invariant under generalized Lorentz transformations (I have used here an imaginary $\mathbf{h}_4$). Einstein's General Relativity shows the egregious *physical* property of being invariant against a change of reference body.

APPENDIX A

Hessenberg's *Vectorial Foundation of Differential Geometry*

Hessenberg introduces an n-leg, $\underline{p}_j$ $(j, k = 1, \ldots, n)$, into every point of a n-dimensional Riemannian manifold. I simplify his representation by taking these n-legs to be orthonormal. Then all indices can be kept downstairs. I refer to equations in his paper by putting them into square brackets "[..]". This gives his equation [24]

$$\underline{p}_j \cdot \underline{p}_k = \delta_{jk} \,. \tag{1}$$

He defines the differential one-form db_{jk} in [39] by (I lower his indices according to footnote on his page 198)

$$d\underline{p}_j \cdot \underline{p}_k \equiv db_{jk} \tag{2}$$

skew-symmetric in indices j and k. His equation [41] gives

$$db_{jk} + db_{kj} = 0 \,. \tag{3}$$

Hessenberg calls the differential one-form db_{jk} the "Orientation Tensor". Nowadays one would write

$$d\underline{p}_j \; = \; \omega_{jl}\,\underline{p}_l\,, \quad d\underline{p}_j \cdot \underline{p}_k \; = \; \omega_{jk}\,, \quad \omega_{jk} + \omega_{kj} = 0 \qquad (2',3')$$

where ω_{jk} is now the connection one-form. We thus have to identify db_{jk} with ω_{jk}. He next introduces in [47] a cogredient differential $d\underline{A}$ of a tensor A with α indices

$$\underline{A} \; = \; A_{j_1 \ldots j_\alpha}\,\underline{p}_{j_1} \cdots \underline{p}_{j_\alpha} \tag{4}$$

in terms of the covariant differentials $\delta A_{j_1 \ldots j_\alpha}$

$$d\underline{A} \; = \; \delta A_{j_1 \ldots j_\alpha}\,\underline{p}_{j_1} \cdots \underline{p}_{j_\alpha} \tag{5}$$

where this differential is defined by [48]

$$\delta A_{j_1 \ldots j_\alpha} \; = \; dA_{j_1 \ldots j_\alpha} \; - \; db_{j_1 k}\,A_{k \ldots j_\alpha} \; - \; \ldots \; - \; db_{j_\alpha k}A_{j_1 \ldots k}\,. \tag{6}$$

In section 20 on page 205 he introduces differential one-forms ω_j that give the Riemannian metric

$$ds^2 \; = \; \omega_j\,\omega_j\,. \tag{7}$$

These differential forms ω_j are his $u^{j\rho}\,dt_\rho$ (and confusingly also denoted as du^j). They are dual to his orthonormal vectors $\underline{p}_k$

$$\omega_j(\underline{p}_k) \; = \; \delta_{jk}\,. \tag{8}$$

Hessenberg's equation [87] is the necessary and sufficient condition that all straightest lines are geodesics in his more general geometry. In this case the connection form specializes to ω'_{jk} obeying what appears now as the first Cartan structural equation for vanishing torsion

$$0 \; = \; -\,d\omega_j \; + \; \omega'_{jk} \wedge \omega_k\,. \tag{9}$$

His notation "$D_{12}\, u^j$" precisely explained in the first footnote on page 211 with the opposite sign against Cartan's "d"-operator. In this case Hessenberg's one-form db_{jk} specializes into the Levi-Civita connection form ω'_{jk}. On the other hand Hessenberg's equation [94] reads now

$$-\, d\omega_j \;=\; \frac{1}{2} U_{ljm}\, \omega_l \wedge \omega_m\,, \quad U_{ljm} \;=\; -\, U_{mjl}\,. \qquad (10)$$

This equation according to Cartan's first structural equation defines the right-hand side of (10) as the negative of the torsion form Θ_j for a vanishing connection that defines the teleparallelism of Hessenberg's straightest lines

$$\frac{1}{2} U_{ljm}\, \omega_l \wedge \omega_m \;=\; -\, \Theta_j \;=\; -\, d\omega_j \,+\, \omega_{jk}\,, \quad \omega_{jk} \;=\; 0\,. \quad (11)$$

Comparing now equations (9) and (10) he needs the development of the Levi-Civita connection form ω'_{jl} in terms of the ω_m. This gives the Ricci rotation coefficients h_{jlm}

$$\omega'_{jl} \;=\; h_{jlm}\omega_m\,, \quad h_{jlm} \;=\; -\, h_{ljm}\,. \qquad (12)$$

This is the meaning of [97] where the Christoffel symbol of the second kind vanishes because of our simplification (1). From (9) and (10) follows now

$$0 \;=\; \frac{1}{2} U_{ljm}\, \omega_l \wedge \omega_m \,+\, h_{ljm}\, \omega_l \wedge \omega_m\,, \qquad (13)$$

or simply Hessenberg's equation [98]

$$0 \;=\; U_{ljm} \,+\, h_{ljm} \,-\, h_{mjl}\,. \qquad (14)$$

By cyclic interchange of the indices one obtains the equations

$$0 \;=\; U_{jml} \,+\, h_{jml} \,-\, h_{lmj} \qquad (15)$$

and

$$0 = U_{mlj} + h_{mlj} - h_{jlm}. \tag{16}$$

Then (14) + (15) - (16) gives

$$2\, h_{mjl} = U_{ljm} + U_{jml} - U_{mlj}. \tag{17}$$

Changing to the indices used in Hessenberg's second footnote on page 211 one obtains (I replace the index "i" by "j" because Microsoft insists on capitalizing it when it comes first)

$$2\, h_{jlk} = U_{ljk} + U_{klj} - U_{jkl}. \tag{18}$$

This does not agree with the expression in Hessenberg's second footnote on page 211. The reason is that he re-defines U by turning its upper index into its first lower index instead of into its second as he stated as a general rule in the footnote on page 198. The interchange of the first two indices in U turns (18) into U', skew-symmetric in its last two indices,

$$2\, h_{jlk} = U'_{jlk} + U'_{lkj} - U'_{kjl}. \tag{19}$$

This agrees with Hessenberg. With U'_{jlk} being the negative of the torsion tensor the tensor h_{jlk} becomes now the negative of the contorsion tensor. The contorsion tensor g_{lkj} becomes by cyclic permutation of the indices

$$2\, g_{jlk} = C_{lkj} + C_{kjl} - C_{jlk}, \tag{20}$$

where the tensor C_{lkj} is skew-symmetric in its first two indices. With

$$C_{lkj} = -U'_{jlk}, \tag{21}$$

comparing (19) and (20) gives

$$h_{jlk} = -g_{jlk}, \tag{22}$$

identifying the contortion tensor with the negative Ricci rotation coefficients of Christoffel's covariant derivative (which became called the Levi-Civita connection). In this way Hessenberg discovered a special case of torsion, namely, as we would nowadays say, a case where the symmetric part of the connection coefficients vanishes, a case of tele-parallelism. The fact that he has a geometric interpretation for it in terms of auto-parallel curves shows that he is writing about a geometric phenomenon, the discrepancy between the straightest and the shortest curves in a geometry with torsion. This is exactly the example that Cartan used to explain his geometry to Einstein pointing out the distinction between rhumbs and geodesics on the sphere.

Coda